Circular flow number of highly edge connected signed graphs

Xuding Zhu *

2012.11.11

Abstract

This paper proves that for any positive integer k, every essentially $(2k+1)$-unbalanced $(12k-1)$-edge connected signed graph has circular flow number at most $2+\frac{1}{k}$.

1 Introduction

Suppose G is a graph. A *circulation* in G is an orientation D of G together with a mapping $f : E(G) \to \mathbb{R}$. A circulation in G can be denoted by a pair (D, f). However, for simplicity, we usually denote it by f, and call D the *orientation associated with* f. The *boundary* of a circulation f is the map $\partial f : V(G) \to \mathbb{R}$ defined as

$$\partial f(v) = \sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e). \qquad (1)$$

Here $E_D^+(v)$ (resp. $E_D^-(v)$) is the set of directed edges in D of the form (v,u) (resp. of the form (u,v)).

A *flow in* G is a circulation in G with $\partial f = 0$. If r is a real number and f is a flow with $1 \le |f(e)| \le r-1$ for every edge e, then f is called a *circular r-flow* in G. The *circular flow number* $\Phi_c(G)$ of G is the least r such that G admits a circular r-flow. If r is an integer and $1 \le |f(e)| \le r-1$ are integers then f is called a *nowhere zero r-flow*. The *flow number* $\Phi(G)$ of G is the least integer r such that G admits a nowhere zero r-flow. It is known [2] that $\Phi(G) = \lceil \Phi_c(G) \rceil$ for any bridgeless graph G.

Integer flow was originally introduced by Tutte [11, 12] as a generalization of map colouring. Tutte proposed the following three conjectures that motivated most of the studies on integer flow in graphs.

- **5-Flow Conjecture**: Every bridgeless graph admits a nowhere zero 5-flow.

*Department of Mathematics, Zhejiang Normal University, China. Grant numbers: NSF11171310 and ZJNSF Z6110786. Email: xudingzhu@gmail.com.

- **4-Flow Conjecture**: Every bridgeless graph with no Petersen minor admits a nowhere zero 4-flow.
- **3-Flow Conjecture**: Every 4-edge connected graph admits a nowhere zero 3-flow.

The concept of circular flow number was introduced in [2] in 1998 as the dual of the circular chromatic number (cf. [15]), and as a refinement of the flow number. However, in early 1980's, Jaeger [3] already studied circular flow in graphs and proposed a conjecture which is equivalent to the following:

- **$(2+\frac{1}{k})$-flow conjecture**: For any positive integer k, if G is $4k$-edge connected, then $\Phi_c(G) \leq 2 + \frac{1}{k}$.

Jaeger's conjecture is very strong. The $k = 1$ case is the 3-flow conjecture, and the $k = 2$ case implies the 5-flow conjecture.

All the above conjectures remain open. Recently, Thomassen [10] made a breakthrough in the study of 3-flow conjecture by proving that every 8-edge connected graph admits a nowhere zero 3-flow. Moreover, for $k \geq 1$, every $(8k^2 + 10k + 3)$-edge connected graph has circular flow number at most $2 + \frac{1}{k}$. This result is improved by Lovász, Thomassen, Wu and Zhang in [7], where it is proved that for any positive integer k, if a graph G has odd edge connectivity at least $6k + 1$, then $\Phi_c(G) \leq 2 + \frac{1}{k}$.

This paper proves an analog of this result for signed graphs.

A *signed graph* is a pair (G, σ), where G is a graph and $\sigma : E(G) \to \{1, -1\}$ assigns to each edge a sign: an edge e is either a *positive edge* (i.e., $\sigma(e) = 1$) or a *negative edge* (i.e., $\sigma(e) = -1$). An *orientation* τ of (G, σ) assigns "orientations" to the edges of G as follows: if $e = xy$ is a positive edge, then the edge is oriented either from x to y or from y to x. In the former case, $e \in E^+_\tau(x) \cap E^-_\tau(y)$, and in the later case, $e \in E^-_\tau(x) \cap E^+_\tau(y)$. If $e = xy$ is a negative edge, then the edge is oriented either from both x and y or towards both x and y. In the former case, $e \in E^+_\tau(x) \cap E^+_\tau(y)$ and e is called a *sink edge*. In the later case, $e \in E^-_\tau(x) \cap E^-_\tau(y)$ and e is called a *source edge*. An orientation τ of (G, σ) may be viewed as a mapping which assigns to each positive edge one of its end vertices as the *head* of the directed edge, and labels each negative edge either as a source edge or as a sink edge. If the orientation is clear from the context, we write $E^+(x)$ for $E^+_\tau(x)$, and etc. An oriented signed graph (i.e., a signed graph plus an orientation) is called a *bidirected graph*.

If all the edges of G are positive, then an orientation of (G, σ) is a directed graph. So the concept of signed graphs is a generalization of graphs, and bidirected graphs is a generalization of digraphs. All the concepts concerning flow in graphs can be naturally extended to signed graphs. For example, a circulation in a signed graph (G, σ) is an orientation τ of (G, σ) together with a mapping $f : E(G) \to \mathbb{R}$. Similarly, we usually denote a circulation in (G, σ) by f, and call τ the orientation of (G, σ) associated with f. The boundary ∂f of a circulation of (G, σ) is defined in the same way as in (1). The concepts of flow, circular r-flow, nowhere zero r-flow, circular flow number and flow number are

extended to signed graphs in the same way. In case a signed graph (G, σ) does not admit a nowhere zero k-flow for any k, then let $\Phi(G, \sigma) = \Phi_c(G, \sigma) = \infty$.

For every question concerning flow in graphs, one can ask the corresponding question for signed graphs. However, less results are known for flow in signed graphs. There are also less conjectures for flow in signed graphs. The only well-known conjecture for flow in signed graphs is the following conjecture proposed by Bouchet [1].

- 6**-Flow Conjecture**: If a signed graph admits a nowhere zero flow, then it admits a nowhere zero 6-flow.

Bouchet [1] proved that if a signed graph admits a nowhere zero flow, then its flow number is at most 216. Zýka [17] improved the upper bound to 30. Khelladi [5] proved that for 4-edge connected graphs, the upper bound can be reduced to 18, and Xu and Zhang [13] proved that for 6-edge connected graphs, the upper bound can be reduced to 6.

The above mentioned results show that highly edge connected graphs have circular flow number close to 2. One naturally wonder if a similar result holds for signed graphs. By applying the main result in [7], we show that the answer to this question is positive, provided that an additional minor necessary condition is satisfied.

If (G, σ) is a signed graph and v is a vertex of G, then by a *switching* at v, we obtain another signed graph (G, σ'), where $\sigma'(e) = -\sigma(e)$ if $e \in E(v)$ and $\sigma'(e) = \sigma(e)$ otherwise. Here $E(v)$ is the set of edges incident to v. We say two signed graphs (G, σ) and (G, σ') with the same underlying graph G are *equivalent* if one can be obtained from the other by a sequence of switchings. It is easy to see that equivalent signed graphs have the same flow number and the same circular flow number. We say a signed graph (G, σ) is $(2k+1)$-*unbalanaced* if every signed graph equivalent to (G, σ) has at least $2k+1$ negative edges, and say (G, σ) is *essentially* $(2k+1)$-*unbalanced* if every signed graph equivalent to (G, σ) has either an even number of negative edges or at least $2k+1$ negative edges. For a signed graph to have circular flow number at most $2 + \frac{1}{k}$, being highly edge connected is not sufficient. For example, a highly edge connected signed graph may have exactly one negative edge. In this case it does not admit a nowhere zero flow. It is also easy to verify that if a signed graph has exactly $2k+1$ negative edges, then its circular flow number is at least $2 + \frac{1}{k}$. So the "correct" question is whether an essentially $(2k+1)$-unbalanced (for some large integer k) highly edge connected signed graph has circular flow number close to 2. Raspaud and Zhu [9] proved that 6-edge connected 3-unbalanced signed graph has flow number at most 4 and has circular flow number strictly less than 4. In this paper, we prove the following result.

Theorem 1 *Let k be a positive integer. If a signed graph (G, σ) is $(12k-1)$-edge connected and essentially $(2k+1)$-unbalanced, then $\Phi_c(G, \sigma) \le 2 + \frac{1}{k}$.*

2 Modulo $(2k+1)$-orientations and Z_{2k+1}-flow

Given an orientation τ of a signed graph (G,σ), the out-degree and in-degree of each vertex x is defined in the same way as for digraphs: $d^+_\tau(x) = |E^+_\tau(x)|$ and $d^-_\tau(x) = |E^-_\tau(x)|$.

Fix a positive integer k. Given a mapping $\beta : V(G) \to Z_{2k+1}$, an orientation τ is called a β-orientation if for every vertex x of G,

$$d^+_\tau(x) - d^-_\tau(x) \equiv \beta(x) \pmod{2k+1}.$$

If $\beta(x) = 0$, then a β-orientation of (G,σ) is called a *modulo* $(2k+1)$*-orientation.*

A mapping $\beta : V(G) \to Z_{2k+1}$ is called a Z_{2k+1}-boundary if $\sum_{x\in V} \beta(x) \equiv 0 \pmod{2k+1}$. Note that for any orientation D of an ordinary graph G, $\sum_{x\in V(G)}(d^+_D(x) - d^-_D(x)) = 0$. Thus for a graph G to have a β-orientation, one necessary condition is that β be a Z_{2k+1}-boundary. The following result, proved in [7], says that if G is highly edge connected, then the converse is also true.

Theorem 2 *[7] For any positive integer k, every $6k$-edge connected graph has a β-orientation for any Z_{2k+1}-boundary β of G.*

First we prove an analog of Theorem 2 for signed graphs. Note that for an orientation τ of a signed graph (G,σ), the summation $\sum_{x\in V(G)}(d^+_\tau(x) - d^-_\tau(x))$ is not necessarily 0.

Theorem 3 *For any positive integer k, for any mapping $\beta : V(G) \to Z_{2k+1}$, every $(2k+1)$-unbalanced $(12k-1)$-edge connected signed graph (G,σ) has a β-orientation.*

Proof. Assume (G,σ) is a $(2k+1)$-unbalanced $(12k-1)$-edge connected signed graph, and β is a mapping from $V(G)$ to Z_{2k+1}. We may assume (G,σ) has the minimum number of negative edges among all signed graphs equivalent to (G,σ). Otherwise let (G,σ') be the one with minimum number of negative edges. Assume (G,σ') is obtained from (G,σ) by switching vertices in X. Let $\beta' : V(G) \to Z_{2k+1}$ be defined as

$$\beta'(v) = \begin{cases} -\beta(v), & \text{if } v \in X, \\ \beta(v), & \text{otherwise.} \end{cases}$$

We shall show that (G,σ') has a β'-orientation, which is equivalent to (G,σ) has a β-orientation.

Fix a mapping $\beta : V \to Z_{2k+1}$. Let $0 \le t \le 2k$ be the integer such that

$$\sum_{x\in V(G)} \beta(x) \equiv 2t \pmod{2k+1}.$$

Let Q be the set of negative edges in (G,σ). If $t \equiv |Q| \pmod 2$, then let τ be an orientation of Q with $(|Q|+t)/2$ sink edges, and $(|Q|-t)/2$ source edges. Let

$$\beta'(x) \equiv \beta(x) - (d^+_\tau(x) - d^-_\tau(x)) \pmod{2k+1}.$$

Since each sink edge (resp. source edge) contributes 2 (resp. -2) to the summation $\sum_{x\in V(G)}(d^+_\tau(x) - d^-_\tau(x))$, we conclude that

$$\begin{aligned}\sum_{x\in V(G)} (d^+_\tau(x) - d^-_\tau(x)) &= (|Q| + t) - (|Q| - t) \\ &= 2t \equiv \sum_{x\in V(G)} \beta(x) \pmod{2k+1}.\end{aligned}$$

Thus

$$\sum_{x\in V(G)} \beta'(x) \equiv \sum_{x\in V(G)} \beta(x) - \sum_{x\in V(G)} (d^+_\tau(x) - d^-_\tau(x)) \pmod{2k+1} \equiv 0 \pmod{2k+1},$$

i.e., β' is a Z_{2k+1}-boundary of G.

If $t \equiv |Q| + 1 \pmod 2$, then let τ be an orientation of Q with $(|Q| + t - 2k - 1)/2$ sink edges and $(|Q| + 2k + 1 - t)/2$ source edges. The same calculation shows that β' is a Z_{2k+1}-boundary of G.

Let R be the subgraph of G induced by positive edges of (G, σ).

Claim 1 *The graph R is $6k$-edge connected.*

Proof. Assume to the contrary that R has an edge cut $E_R[X, \bar{X}]$ of size at most $6k - 1$. Since G is $(12k - 1)$-edge connected, we have $|E_Q[X, \bar{X}]| \geq 6k > |E_R[X, \bar{X}]|$. Let (G, σ') obtained from (G, σ) by switching at all vertices in X. Then $\sigma'(e) = -\sigma(e)$ if $e \in E_G[X, \bar{X}]$ and $\sigma'(e) = \sigma(e)$ otherwise. Thus (G, σ') has less negative edges than (G, σ), contrary to our choice of (G, σ). ∎

By Theorem 2, R has a β'-orientation D, i.e., $d^+_D(x) - d^-_D(x) \equiv \beta'(x) \pmod{2k+1}$ for every vertex x. The union of this orientation of R and the previously chosen orientation τ of Q is an orientation τ' of (G, σ), with $d^+_{\tau'}(x) = d^+_\tau(x) + d^+_D(x)$ and $d^-_{\tau'}(x) = d^-_\tau(x) + d^-_D(x)$. Hence for every vertex x,

$$\begin{aligned} d^+_{\tau'}(x) - d^-_{\tau'}(x) &= (d^+_D(x) - d^-_D(x)) + (d^+_\tau(x) - d^-_\tau(x)) \\ &\equiv \beta'(x) + (d^+_\tau(x) - d^-_\tau(x)) \pmod{2k+1} \\ &\equiv \beta(x) \pmod{2k+1}\end{aligned}$$

I.e., τ' is a β-orientation of (G, σ). ∎

For Theorem 3, the condition that (G, σ) be $(2k + 1)$-unbalanced is needed. For example, if β is not a Z_{2k+1}-boundary, and (G, σ) has no negative edges, then (G, σ) cannot have a β-orientation. For the proof above, if $|Q| < 2k + 1$, then the numbers $|Q| - t$ and $|Q| + t - (2k + 1)$ appeared in the proof might be negative. On the other hand, if we restrict to modulo $(2k + 1)$-orientations, this condition can be slightly weakened. The following Theorem can be proved in the same way as Theorem 3.

Theorem 4 *For any positive integer k, every $(12k - 1)$-edge connected, essentially $(2k+1)$-unbalanced signed graph (G, σ) has a modulo $(2k+1)$-orientation.* ∎

Assume (G, σ) is a signed graph and A is an abelian group. An A-circulation of (G, τ) is an orientation τ of (G, σ) together with a mapping $f : E(G) \to A$. The boundary of an A-circulation f of (G, σ) is defined in the same way as before, i.e., $\partial f(x) = \sum_{e \in E^+(x)} f(e) - \sum_{e \in E^-(x)} f(e)$, where the summation is the group operation. Similarly, an A-flow of (G, σ) is an A-circulation f with $\partial f(x) = 0$ for every vertex x.

Let k be a positive integer. We consider the group Z_{2k+1}. A *special Z_{2k+1}-circulation* (resp. *special Z_{2k+1}-flow*) in a signed graph (G, σ) is a Z_{2k+1}-circulation (resp. a Z_{2k+1}-flow) f with $f(e) \in \{k, k+1\}$ for every edge e.

Corollary 1 *For any positive integer k, every $(12k-1)$-edge connected, essentially $(2k+1)$-unbalanced signed graph (G, σ) admits a special Z_{2k+1}-flow.*

Proof. By Theorem 4, (G, σ) has a modulo $(2k+1)$-orientation τ. Let $f(e) = k$ for all edges e of G. Then f is a special Z_{2k+1}-flow in (G, σ). ∎

3 Circular flow number

Assume $p \geq 2q$ are positive integers. A (p, q)-flow in a signed graph (G, σ) is an integer flow f with $f(e) \in \{q, q+1, \ldots, p-q\}$ for every edge e. If $(G.\sigma)$ admits a (p, q)-flow f, then $g(e) = \frac{f(e)}{q}$ is a circular p/q-flow in (G, σ). Hence $\Phi_c(G, \sigma) \leq p/q$. (The converse is also true: if $\Phi_c(G, \sigma) \leq p/q$, then (G, σ) admits a (p, q)-flow. But we shall not use that.) Thus to prove $\Phi_c(G, \sigma) \leq 2 + \frac{1}{k}$, it suffices to prove that (G, σ) admits a $(2k+1, k)$-flow.

It is well-known (cf. [14]) that an ordinary graph G admits a $(2k+1, k)$-flow if and only if G admits a special Z_{2k+1}-flow. However, this is not the case for signed graphs. It was proved in [13] that if (G, σ) is a cubic signed graph which admits a special Z_3-flow, then (G, σ) admits a nowhere zero 3-flow (or equivalently a $(3, 1)$-flow) if and only if G has a perfect matching. Nevertheless, we shall prove that the condition of Corollary 1 implies that (G, σ) admits a $(2k+1, k)$-flow and hence (G, σ) has circular flow number at most $2 + \frac{1}{k}$.

Theorem 5 *If a signed graph (G, σ) is $(12k-1)$-edge connected and essentially $(2k+1)$-unbalanced, then (G, σ) admits a $(2k+1, k)$-flow.*

Proof. Assume (G, σ) is $(12k-1)$-edge connected and essentially $(2k+1)$-unbalanced. We assume that (G, σ) has the minimum number of negative edges among all signed graphs equivalent to (G, σ). Let R and Q be the subgraphs of (G, σ) induced by the set of positive edges and by the set of negative edges, respectively.

We shall construct a $(2k+1, k)$-flow in (G, σ). This is done in two steps. In the first step, we construct a special Z_{2k+1}-circulation f in the subgraph Q. In the second step, we construct a special Z_{2k+1}-circulation g in R, so that $f + g$ is a $(2k+1, k)$-flow. In taking the sum $f + g$, we view f (resp. g) as a circulation in (G, σ) with $f(e) = 0$ for every positive edge e (resp. $g(e) = 0$ for every negative edge e).

Given a special Z_{2k+1}-circulation f in Q. For a subset X of $V(G)$. Let $E_G[X,\bar{X}]$ is the set of edges in G with one end vertex in X and the other in $\bar{X} = V \setminus X$ (we write $E[X,\bar{X}]$ for short, if the graph G is clear from the context). For example, $E_R[X,\bar{X}] = E_G[X,\bar{X}] \cap R$. Let

$$\begin{aligned} \partial f(X) &= \sum_{v \in X} \partial f(v), \\ \Theta(X) &= k|E_R(X,\bar{X}]| + \partial f(X). \end{aligned}$$

We say the circulation f is *balanced* if the following hold:

- $\sum_{x \in V(G)} \partial f(x) = 0$.
- For any subset X of V, $\Theta(X) \geq k-2$.

The special Z_{2k+1}-circulation f in Q we construct in the first step will be a balanced circulation. Lemma 1 below shows that such a circulation exists.

Lemma 1 *There exists a balanced special Z_{2k+1}-circulation f in Q.*

Proof. By Claim 1, R is $6k$-edge connected. By Nash-Williams' Theorem (cf. [14]), R contains $3k$ edge disjoint spanning trees, $T_1, T_2, \ldots, T_{3k}$. Let G' be the subgraph of G induced by $Q \cup T_1 \cup T_2$ (here G' is an ordinary graph, the signs on the edges are ignored). It is well-known (cf. [14]) that any spanning tree of G' contains a *parity subgraph* F of G', i.e., for each vertex x,

$$d_F(x) \equiv d_{G'}(x) \pmod 2.$$

Let F be a parity subgraph of G' contained in T_2. Then $G' - F$ is connected (as it contains a spanning tree T_1) and every vertex has an even degree, and hence has an eulerian cycle W. We orient the edges in Q as follows: We start from an arbitrary vertex v_0, follow the eulerian cycle W, label the edges in Q alternately source edge and sink edge (the positive edges in W are ignored in this labeling process). In other word, assume we traverse the eulerian cycle W, the negative edges encountered on the way are $e_1, e_2, \ldots, e_q$. Then e_{2i-1} are source edges and e_{2i} are sink edges. In particular, if $|Q|$ is even, then the number of sink edges is the same as the number of source edges. If $|Q|$ is odd, then the number of sink edges is 1 less than the number of source edges. This completes the construction of the orientation τ of Q.

If $|Q|$ is even, then let $f(e) = k$ for every edge $e \in Q$. If $|Q|$ is odd, then $|Q| \geq 2k+1$. Let T' be a set of k sink edges. Let $f(e) = k+1$ if $e \in T'$ and $f(e) = k$ if $e \in Q \setminus T'$. This completes the construction of the special Z_{2k+1}-circulation f in Q.

Now we prove that f is a balanced special Z_{2k+1}-circulation.

It follows from the construction that $\sum_{e \in S} f(e) = \sum_{e \in T} f(e)$. As each sink (resp. source) edge contributes $2f(e)$ (resp. $-2f(e)$) to $\sum_{x \in V} \partial f(x)$, we have

$$\sum_{x \in V} \partial f(x) = 0.$$

We shall show that for any subset X of V, $\Theta(X) \geq k-2$. Starting from a vertex in $\bar{X}$, we traverse the eulerian cycle W. Each time we enter X and leave X, we traverse through a segment of W contained in X, and two edges in $E[X,\bar{X}]$. Let $(e'_1, e'_2, \ldots, e'_b)$ be such a segment, with $e'_1, e'_b \in E[X,\bar{X}]$, and $e'_i \in G[X]$ for $i = 2, 3, \ldots, b-1$. Now we calculate the contribution of these edges to $\Theta(X)$. The following follows from the definition of $\Theta(X)$.

1. If $e'_i \in G[X]$ is a source (resp. sink) edge, then e'_i contributes $-2f(e'_i) = -2k$ (resp. $2f(e'_i)) \geq 2k$) to $\Theta(X)$.

2. If $e'_i \in G[X]$ is a positive edge, then e'_i contributes 0 to $\Theta(X)$.

3. If $i \in \{1, b\}$ and e'_i is a source (resp. sink) edge, then e'_i contributes $-k$ (resp. $f(e'_i) \geq k$) to $\Theta(X)$.

4. If $i \in \{1, b\}$ and e'_i is a positive edge, then it contributes k to $\Theta(X)$.

By our orientation, the negative edges in W are alternately source edge and sink edge, except that in case $|Q|$ is odd, there are two consecutive source edges (i.e., two source edges not separated by a sink edge, but maybe separated by some positive edges).

We claim that the contribution of the edges in this segment to $\Theta(X)$ is non-negative, except that when the segment contains two consecutive source edges, the contribution of this segment is at least $-2k$.

For the proof of this claim, we need to consider a few cases according to whether e'_1, e'_b are positive edges, or one positive and the other is a source edge, or one is positive and the other is a sink edge, etc. However, each case is straightforward. We just consider two cases, and for simplicity, we assume that the segment does not contain two consecutive source edges (if it does have two consecutive source edges, then we need to add $-2k$ to the total contribution).

If both e'_1, e'_b are positive edges, then the number of source edges in this segment is at most one more than the number of sink edges. Since each of e'_1, e'_b contributes k to $\Theta(X)$, we conclude that the total contribution of this segment to $\Theta(X)$ is non-negative.

If both e'_1 and e'_b are sink edges, the number of source edges in this segment is one more than the number of sink edges. Since each of e'_1, e'_b contributes at least k to $\Theta(X)$, we conclude that the total contribution of this segment to $\Theta(X)$ is non-negative.

Add up the contribution of all the edges in the eulerian cycle W, we conclude that the total contribution is at most $-2k$, where $-2k$ is contributed by the segment containing two consecutive source edges.

Now each spanning tree T_i for $i = 3, 4, \ldots, 3k$ contains at least one edge in $E_R(X, \bar{X}]$, and hence contributes at least k to $\Theta(X)$. So $\Theta(X) \geq 3k - 2 - 2k = k - 2$. This completes the proof of Lemma 1. ∎

Let f be a balanced special Z_{2k+1}-circulation f in Q.

Claim 2 *There is a special $(2k+1)$-circulation g of R such that $f + g$ is a special Z_{2k+1}-flow in (G, σ).*

Proof. Let $\beta : V(G) \to Z_{2k+1}$ be defined as

$$\beta(x) \equiv 2\partial f(x) \pmod{2k+1}.$$

Then $\sum_{x\in V(G)} \beta(x) \equiv 0 \pmod{2k+1}$.

Since R is $6k$-edge connected, by Theorem 3, R has a β-orientation D. Let $g(e) = k$ for $e \in R$. Then for each vertex x,

$$\begin{aligned}\partial g(x) &\equiv k\beta(x) \pmod{2k+1}\\ &\equiv 2k\partial f(x) \pmod{2k+1}\\ &\equiv -\partial f(x) \pmod{2k+1}.\end{aligned}$$

Hence $\partial(f + g)(x) = \partial f(x) + \partial g(x) \equiv 0 \pmod{2k+1}$ for each vertex x, i.e., $f + g$ is a special Z_{2k+1}-flow in (G, σ). ∎

For a Z_{2k+1}-flow ϕ in (G, σ), let

$$||\phi|| = \sum_{x\in V(G)} |\partial\phi(x)|.$$

Thus a special Z_{2k+1}-flow ϕ is a $(2k + 1, k)$-flow if and only if $||\phi|| = 0$.

Among all the special Z_{2k+1}-flows ϕ in (G, σ) of the form $f + g$, choose one for which $||(f+g)||$ is minimum. If $||(f+g)|| = 0$, then $f+g$ is a $(2k+1, k)$-flow in (G, σ), and we are done.

Assume this is not the case. Let $V^+ = \{x : \partial(f + g)(x) > 0\}$ and $V^- = \{x : \partial(f + g)(x) < 0\}$. Since $\sum_{x\in V(G)} \partial(f + g)(x) = 0$, $V^+ \neq \emptyset$ and $V^- \neq \emptyset$. Let D be the orientation of R associated with the circulation g. We say a vertex y of G is *reachable* if there is a directed path in D from y to a vertex $x \in V^-$. In particular, every vertex in V^- is reachable. Let Y be the set of all reachable vertices.

Assume first that $Y \cap V^+ \neq \emptyset$. Let P be a directed path in D from $y \in V^+$ to $x \in V^-$. We reverse the orientation of the edges in P, and let $g'(e) = 2k+1-g(e)$ for $e \in P$ and $g'(e) = g(e)$ for $e \notin P$. Then $f + g'$ is a special Z_{2k+1}-flow in (G, σ) with

$$\partial(f + g')(v) = \begin{cases}\partial(f + g)(v) - (2k + 1), & \text{if } v = y,\\ \partial(f + g)(v) + (2k + 1), & \text{if } v = x,\\ \partial(f + g)(v), & \text{otherwise.}\end{cases}$$

As $f + g$ is a Z_{2k+1}-flow, $\partial(f + g)(x) < 0$ and $\partial(f + g)(y) > 0$ imply that $\partial(f+g)(x) = -a(2k+1)$ for some positive integer a, and $\partial(f+g)(y) = b(2k+1)$ for some positive integer b. Therefore $||(f+g')|| = ||(f+g)||-2(2k+1)$, contrary to our choice of g.

Assume $Y \cap V^+ = \emptyset$. Since $X^- \subseteq Y$, we have $\sum_{v\in Y} \partial(f+g)(v) \leq -(2k+1)$. If there exist $y' \in \bar{Y}$ and $y \in Y$ such that (y', y) is a directed edge of D, then y' would be a reachable vertex, a contradiction. Thus all edges in $E_D[Y, \bar{Y}]$ are oriented from Y to $\bar{Y}$. Observe that $\sum_{v\in Y} \partial(f + g)(v) = \sum_{v\in Y} \partial f(v) +$

$\sum_{v\in Y} \partial g(v)$. Since each edge in $E_D[Y, \bar{Y}]$ contribute k to $\sum_{v\in Y} \partial g(v)$, and every other edge contributes 0 to $\sum_{v\in Y} \partial g(v)$, we conclude that $\sum_{v\in Y} \partial(f + g)(v) = \Theta(Y)$. By Lemma 1, $\Theta(Y) \geq k - 2$, contrary to the conclusion that $\sum_{v\in Y} \partial(f+g)(v) \leq -(2k+1)$.

This completes the proof of Theorem 5. ∎

Jaeger's $(2 + \frac{1}{k})$-flow conjecture is sharp: there are $(4k-1)$-edge connected graphs G for which $\Phi_c(G) > 2 + \frac{1}{k}$.

Corresponding to Jaeger's conjecture, one naturally wonder what is the least integer $\psi(k)$ such that every essentially $(2k+1)$-unbalanced $\psi(k)$-edge connected signed graph (G, σ) have $\Phi_c(G, \sigma) \leq 2 + \frac{1}{k}$? The current known bounds are $4k \leq \psi(k) \leq 12k - 1$. It would be interesting to narrow the gap between the upper and lower bounds.

References

[1] A. Bouchet, *Nowhere-zero integral flows on bidirected graph*, J. Combin. Theory Ser. B 34 (1983), 279-292.

[2] L. A. Goddyn, M. Tarsi and C. Q. Zhang, *On (k, d)-colorings and fractional nowhere zero flows*, J. Graph Theory, 28(1998), 155-161.

[3] F. Jaeger, On circular flows in graphs. Proceedings of the 6th Hungarian Colloquium on Combinatorics, Eger, Hungary, 1981.

[4] F. Jaeger, *Nowhere-zero flow Problems*, Selected topics in Graph Theory 3 Academic Press, London 1988, 71-95.

[5] A. Khelladi, *Nowhere-zero integral chains and flows in bidirected graphs*, J. Combin. Theory Ser. B. 43 (1987), 95-115.

[6] S. Kundu, *Bounds on the number of disjoint spanning trees*, J. Combin. Theory Ser. B. 17 (1974), 199-203.

[7] L. M. Lovasz, C. Thomassen, Y. Wu and C. Q. Zhang, *Nowhere-zero 3-flows and modulo k-orientations*, J. Combin. Theory Ser. B., to appear.

[8] C.S.J.A. Nash-Williams, *Edge disjoint spanning trees of finite graphs*, J. London Math. Soc. 36(1961), 445-450.

[9] A. Raspaud and X. Zhu, *Circular flow on signed graphs*, J. Combin. Theory Ser. B. 101(2011), no. 6, 490-501.

[10] C. Thomassen, *The weak 3-flow conjecture and the waek circular flow conjecture*, J. Combin. Theory Ser. B. 102(2012), 521-529.

[11] W. T. Tutte, *On the embedding of linear graphs in surfaces*, Proc. London Math. Soc., Ser. 2, 51 (1949), 474-483.

[12] W. T. Tutte, *A construction on the theory of chromatic polynomial*, Canad. J. Math., 6 (1954), 80-91.

[13] R. Xu and C. Zhang, *On flows in bidirected graphs*, Discrete Mathematics, 299(2005) 335-343.

[14] C. Q. Zhang, *Integer flows and cycle covers of graphs*, Marcel Dekker, New York. 1997.

[15] X. Zhu, *Circular chromaic number - a survey*, Discrete Mathematics, 229 (2001), no. 1-3, 371-410.

[16] X. Zhu, *Recent developments in circular colouring of graphs* Topics in Discrete Mathematics, 2006, 497-550.

[17] O. Zýka, *Nowhere-zero 30-flow on bidirected graphs*, Thesis, Charles Universiy, Pragua,1 87, KAM-DIMATIA Seris 87-26.